第六次作业参考答案

党金龙

2023 年 12 月 20 日

习题 1

光子与质子相互作用顶点的一般形式可以写为

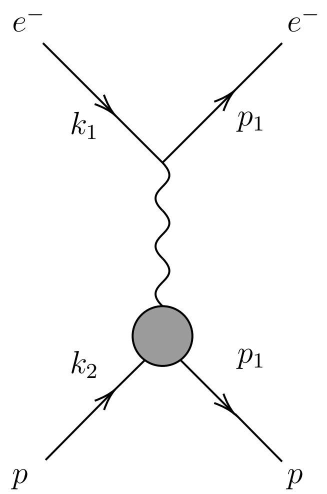
$$\bar{u}\left(p^{\prime}\right)\left[\gamma^{\mu} F_{1}\left(q^{2}\right)+\frac{i \sigma^{\mu \nu} q_{\nu}}{2 m} F_{2}\left(q^{2}\right)\right] u(p) \tag{1}$$

其中 $q=p^{\prime}-p$ 是进入顶点的光子动量, $\sigma^{\mu\nu}=\frac{1}{2} i\left[\gamma^{\mu}, \gamma^{n} u\right]$。利用这个结果，计算电子和质子散射关于散射角的树图微分截面（忽略电子质量)，结果是著名的 Rosenbluth 公式：

$$\frac{d \sigma}{d \cos \theta}=\frac{\pi \alpha^{2}}{2 E^{2}\left[1+\frac{2 E}{m} \sin ^{2} \frac{\theta}{2}\right] \sin ^{4} \frac{\theta}{2}}\left[\left(F_{1}^{2}-\frac{q^{2}}{4 m^{2}} F_{2}^{2}\right) \cos ^{2} \frac{\theta}{2}-\frac{q^{2}}{2 m^{2}}\left(F_{1}+F_{2}\right)^{2} \sin ^{2} \frac{\theta}{2}\right] \tag{2}$$

解：

该过程费曼图可以画为



在初始质子静止的参考系中，动量分量可写做

$$k_{1}=(E, 0,0, E), \quad p_{1}=\left(E^{\prime}, E^{\prime} \sin \theta, 0, E^{\prime} \cos \theta\right), \quad k_{2}=(M, 0,0,0) \tag{3}$$

再依据动量守恒，$k_1+k_2=p_1+p_2$，与在壳条件，$p_2^2=M^2$，可知

$$E^{\prime}=\frac{M E}{M+2 E \sin ^{2} \frac{\theta}{2}} \tag{4}$$

下面采用记号，$q=k_1-p_1$，$t=q^2$，并用 U 表示质子的旋量，M 表示质子的质量。则振幅可以写为

$$\mathrm{i}\mathcal{M}=\bar{U}\left(p_2\right)(+\mathrm{ie})\left[\gamma^\mu F_1\left(q^2\right)+\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2M}F_2\left(q^2\right)\right]U\left(k_2\right)\frac{-\mathrm{i}\eta_{\mu\mu'}}{q^2}\bar{u}\left(p_1\right)(-\mathrm{ie})\gamma^{\mu'}u\left(k_1\right) \tag{5}$$

使用 Gordon identity 得到

$$\mathrm{i}\mathcal{M}=e^2\bar{U}\left(p_2\right)\left[\gamma^\mu\left(F_1+F_2\right)-\frac{\left(p_2+k_2\right)^\mu}{2M}F_2\right]U\left(k_2\right)\frac{-\mathrm{i}}{q^2}\bar{u}\left(p_1\right)\gamma_\mu u\left(k_1\right), \tag{6}$$

则对初态求平均，末态求和得到

$$\begin{aligned}\frac{1}{4}\sum|\mathcal{M}|^2=&\frac{e^4}{4q^4}\operatorname{tr}\left[\left(\gamma^\mu\left(F_1+F_2\right)-\frac{\left(p_2+k_2\right)^\mu}{2M}F_2\right)\left(\not{k}_2+M\right)\right.\\&\left.\times\left(\gamma^\rho\left(F_1+F_2\right)-\frac{\left(p_2+k_2\right)^\rho}{2M}F_2\right)\left(\not{p}_2+M\right)\right]\operatorname{tr}\left[\gamma_\mu\not{k}_1\gamma_\rho\not{p}_1\right]\\=&\frac{4e^4M^2}{q^4}\left[\left(2E^2+2E'^2+q^2\right)\left(F_1+F_2\right)^2\right.\\&\left.-\left(2F_1F_2+F_2^2\left(1+\frac{q^2}{4M^2}\right)\right)\left(\left(E+E'\right)^2+q^2\left(1-\frac{q^2}{4M^2}\right)\right)\right].\end{aligned} \tag{7}$$

利用

$$2F_1F_2+F_2^2\left(1+\frac{q^2}{4M^2}\right)=\left(F_1+F_2\right)^2-F_1^2+\frac{q^2}{4M^2}F_2^2 \tag{8}$$

得到

$$\frac{1}{4}\sum|\mathcal{M}|^2=\frac{4e^4M^2}{q^4}\left[\frac{q^4}{2M^2}\left(F_1+F_2\right)^2+4\left(F_1^2-\frac{q^2}{4M^2}F_2^2\right)EE'\cos^2\frac{\theta}{2}\right] \tag{9}$$

通过

$$E'-E=\frac{q^2}{2m} \tag{10}$$

$$q^2=-4E'E\sin^2\frac{\theta}{2} \tag{11}$$

得到

$$\begin{aligned}\frac{1}{4}\sum|\mathcal{M}|^2=&\frac{16e^4E^2M^3}{q^4\left(M+2E\sin^2\frac{\theta}{2}\right)}\\&\times\left[\left(F_1^2-\frac{q^2}{4M^2}F_2^2\right)\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\left(F_1+F_2\right)^2\sin^2\frac{\theta}{2}\right]\end{aligned} \tag{12}$$

而散射截面对于 $A+B\rightarrow 1+2$ 过程可以写为

$$\mathrm{d}\sigma=\frac{1}{2E_A2E_B\left|\mathbf{v}_A-\mathbf{v}_B\right|}\int\frac{\mathrm{d}^3p_1\ \mathrm{d}^3p_2}{(2\pi)^62E_12E_2}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}\left(p_1+p_2-p_A-p_B\right) \tag{13}$$

代入 $E_A=E$, $E_B=M$, $E_1=E'$, $\left|\mathbf{v}_A-\mathbf{v}_B\right|\approx 1$, 得到

$$\begin{aligned}\mathrm{d}\sigma_{\mathrm{L}}&=\frac{1}{4EM}\int\frac{\mathrm{d}^3p_1\ \mathrm{d}^3p_2}{(2\pi)^62E_12E_2}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}\left(p_1+p_2-p_A-p_B\right)\\&=\frac{1}{4EM}\int\frac{E'^2\ \mathrm{d}E'\mathrm{d}\cos\theta\mathrm{d}\varphi}{(2\pi)^32E'2E_2}|\mathcal{M}|^2(2\pi)\delta\left(E'+E_2-E-M\right)\\&=\frac{1}{4EM}\int\frac{E'^2\ \mathrm{d}E'\mathrm{d}\cos\theta\mathrm{d}\varphi}{(2\pi)^22E'2E_2}|\mathcal{M}|^2\left[1+\frac{E'-E\cos\theta}{E_2\left(E'\right)}\right]^{-1}\delta\left(E'-\frac{ME}{M+2E\sin^2\frac{\theta}{2}}\right)\\&=\frac{1}{4EM}\int\frac{\mathrm{d}\cos\theta}{8\pi}|\mathcal{M}|^2\frac{E'}{M+2E\sin^2\frac{\theta}{2}}\end{aligned} \tag{14}$$

其中 E_2 依赖于 E',

$$E_2 = \sqrt{M^2 + E^2 + E'^2 - 2E'E\cos\theta} \tag{15}$$

于是有

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{1}{32\pi\left(M + 2E\sin^2\frac{\theta}{2}\right)^2}|\mathcal{M}|^2 \tag{16}$$

代入振幅表达式，得到

$$\begin{aligned}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right)_{\mathrm{L}} =& \frac{\pi\alpha^2}{2E^2\left(1 + \frac{2E}{M}\sin^2\frac{\theta}{2}\right)\sin^4\frac{\theta}{2}} \\ &\times\left[\left(F_1^2 - \frac{q^2}{4M^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\left(F_1 + F_2\right)^2\sin^2\frac{\theta}{2}\right].\end{aligned} \tag{17}$$

习题 2

针对重整化的 QED 拉格朗日量：

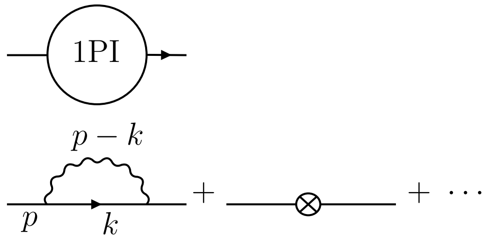
$$\mathcal{L} = -\frac{1}{4}Z_3F_{\mu\nu}F^{\mu\nu} + Z_2\bar{\psi}\left(i\gamma^\mu\partial_\mu - Z_m m\right)\psi + Z_1 e\bar{\psi}\gamma^\mu A_\mu\psi \tag{18}$$

在在壳重整化方案下，用截断正规化（即对欧式圈动量 l_E 加一个紫外截断 Λ ）计算单圈水平的 Z_1 和 Z_2。 Wald 恒等式要求 $Z_1 = Z_2$，请判断在这个正规化下 Wald 恒等式是否被破坏？其原因是什么？

解：

首先在单圈水平计算 Z_2，依据

$-i\Sigma(\not{p}) =$



不妨写为

$$-i\Sigma(\not{p}) = -i\Sigma_2(\not{p}) + i\left(\not{p}\delta_2 - \delta_m\right) \tag{19}$$

其中 $Z_2 = 1 + \delta_2$，而由于在壳重整化条件

$$\left.\frac{d}{d\not{p}}\Sigma(\not{p})\right|_{\not{p}=m} = 0 \tag{20}$$

得到

$$Z_2 = 1 - \delta_2 = 1 - \left.\frac{d}{d\not{p}}\Sigma_2(\not{p})\right|_{\not{p}=m} \tag{21}$$

下面计算 $\Sigma_2(\not{p})$，为避免红外发散，给光子加上一个小质量 μ

$$\begin{aligned}-i\Sigma_2 =& \int\frac{d^4k}{(2\pi)^4}(-ie\gamma^\mu)\frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon}(-ie\gamma^\nu)\frac{-ig_{\mu\nu}}{(p-k)^2 - \mu^2 + i\epsilon} \\ =&(-ie)^2\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^\mu(\not{k} + m)\gamma_\mu}{(k^2 - m^2)[(p-k)^2 - \mu^2]} \\ =& -e^2\int_0^1 dx\int\frac{d^4l}{(2\pi)^4}\frac{-2x\not{p} + 4m}{(l^2 - \Delta + i\epsilon)^2}\end{aligned} \tag{22}$$

其中 $\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m^2$,, 并且转动至欧氏空间 $l_E^0 = -il^0$, 并且加上动量截断 Λ,

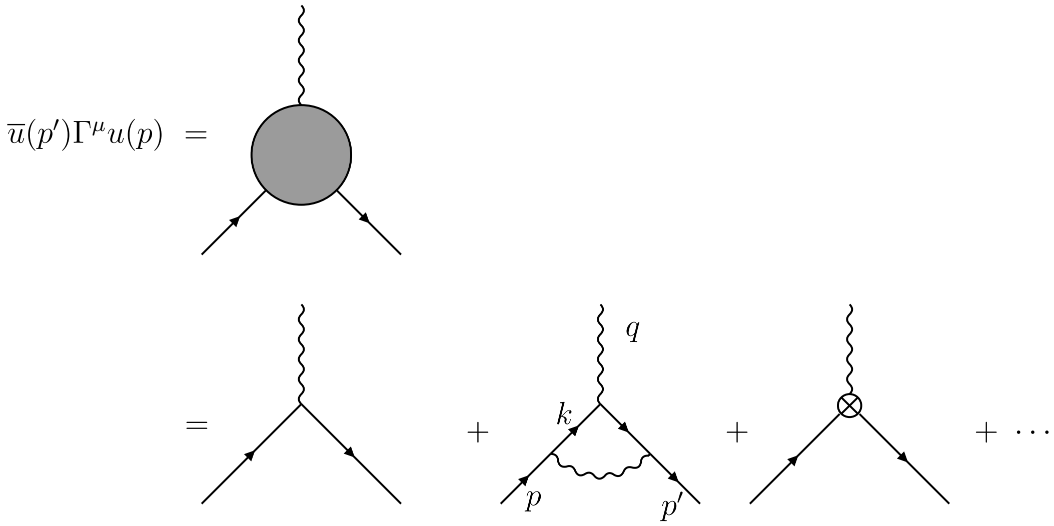
$$\begin{aligned}\Sigma_2 =& -ie^2\int_0^1 dx\int\frac{d^4l_E}{(2\pi)^4}\frac{-2x\not{p}+4m}{(l_E^2+\Delta)^2}\\ =&e^2\int_0^1 dx\int\frac{d\Omega_4}{(2\pi)^4}\int_0^\Lambda dl_E l_E^3\frac{-2x\not{p}+4m}{(l_E^2+\Delta)^2}\\ =&\frac{e^2}{8\pi^2}\int_0^1 dx(-x\not{p}+2m)\left(\ln(1+\frac{\Delta}{\Lambda^2})-\frac{\Lambda^2}{\Delta+\Lambda^2})\right)\end{aligned} \tag{23}$$

所以

$$\begin{aligned}\delta_2 = \frac{\mathrm{d}\Sigma_2(\not{p})}{\mathrm{d}\not{p}}\bigg|_{\not{p}=m} =& \frac{e^2}{8\pi^2}\int_0^1\ \mathrm{d}x x\left[-\ln\left(1+\frac{\Lambda^2}{\Delta_m}\right)+\frac{\Lambda^2}{\Delta_m+\Lambda^2}+\frac{2\Lambda^4 m^2(1-x)(2-x)}{\Delta_m(\Delta_m+\Lambda^2)^2}\right]\\ \xrightarrow{\Gamma\to\infty}& \frac{e^2}{8\pi^2}\int_0^1\ \mathrm{d}x x\left[-\ln\left(1+\frac{\Lambda^2}{\Delta_m}\right)+1+\frac{2m^2(1-x)(2-x)}{\Delta_m}\right]\end{aligned} \tag{24}$$

其中 $\Delta_m = \Delta|_{\not{p}=m} = x\mu^2 + m^2(1-x)^2$。

接下来在单圈水平计算 Z_1，依据



不妨写为

$$\bar{u}(p')(-ie\Gamma^\mu)u(p) = \bar{u}(p')(-ie\gamma^\mu)u(p) + \bar{u}(p')(-ie\delta\Gamma^\mu)u(p) - \bar{u}(p')ie\gamma^\mu\delta_1 u(p) \tag{25}$$

其中 $Z_1 = 1+\delta_1$，而由于在壳重整化条件 $-ie\Gamma^\mu(p'-p) = -ie\gamma^\mu$, 那么,

$$\bar{u}(p')\delta\Gamma^\mu u(p)|_{p'=p} = -\bar{u}(p')\gamma^\mu\delta_1 u(p)|_{p'=p} \tag{26}$$

而一般的，由 Lorentz 结构可以写

$$\bar{u}(p')\delta\Gamma^\mu u(p) = \bar{u}\left(p'\right)\left[\gamma^\mu\delta F_1\left(q^2\right)+\frac{i\sigma^{\mu\nu}q_\nu}{2m}\delta F_2\left(q^2\right)\right]u(p) \tag{27}$$

则

$$\delta_1 = -\delta F_1\left(q^2=0\right) \tag{28}$$

依据 Feynman 规则有

$$\begin{aligned}
\bar{u}(p')\delta\Gamma^\mu u(p) =& \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\nu\rho}}{(k-p)^2-\nu^2+i\epsilon}\bar{u}(p')(-ie\gamma^\nu)\frac{i(\not{k}'+m)}{k^2-m^2+i\epsilon}\gamma^\mu\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}(-ie\gamma^\rho)u(p) \\
=&2ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(p')[\not{k}\gamma^\mu\not{k}'+m^2\gamma^\mu-2m(k+k')^\mu]u(p)}{((k-p)^2-\mu^2+i\epsilon)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)} \\
=&2ie^2\int\frac{d^4l}{(2\pi)^4}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3}\bar{u}(p') \\
&\times[\gamma^\mu(-\frac{1}{2}l^2+(1-x)(1-y)q^2+(1-4z+z^2)m^2)+\frac{i\sigma^{\mu\nu}q_\nu}{2m}(2m^2z(1-z))]u(p)
\end{aligned} \tag{29}$$

其中 $D=l^2-\Delta+i\epsilon$, $\Delta=-xyq^2+(1-z)^2m^2+z\mu^2$，相似地，在动量截断下可以求积分

$$\mathcal{I}_1=\int_\Lambda\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta)^3}=-i\int\frac{\mathrm{d}\Omega_4}{(2\pi)^4}\int_0^\Lambda dkk^3\frac{1}{(k^2+\Delta)^3}=-\frac{\mathrm{i}}{32\pi^2}\frac{\Lambda^4}{\Delta(\Lambda^2+\Delta)^2} \tag{30}$$

$$\mathcal{I}_2=\int_\Lambda\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k^2}{(k^2-\Delta)^3}=i\int\frac{\mathrm{d}\Omega_4}{(2\pi)^4}\int_0^\Lambda dkk^3\frac{k^2}{(k^2+\Delta)^3}=\frac{\mathrm{i}}{16\pi^2}\left[\ln\left(1+\frac{\Lambda^2}{\Delta}\right)+\frac{\Delta(4\Lambda^2+3\Delta)}{2(\Lambda^2+\Delta)^2}-\frac{3}{2}\right] \tag{31}$$

那么，

$$\begin{aligned}
\bar{u}(p')\delta\Gamma^\mu u(p) =&2ie^2\int_0^1 dxdydz\delta(x+y+z-1)2\bar{u}(p') \\
&\times[\gamma^\mu(-\frac{1}{2}\mathcal{I}_2+((1-x)(1-y)q^2+(1-4z+z^2)m^2)\mathcal{I}_1)+\mathcal{I}_1\frac{i\sigma^{\mu\nu}q_\nu}{2m}(2m^2z(1-z))]u(p)
\end{aligned} \tag{32}$$

所以，

$$\begin{aligned}
\delta_1 =& -\delta F_1(q^2=0) \\
=& -4ie^2\int_0^1 dxdydz\delta(x+y+z-1)(-\frac{1}{2}\mathcal{I}_2+((1-4z+z^2)m^2)\mathcal{I}_1)\Big|_{q^2=0} \\
=&\frac{e^2}{8\pi^2}\int_0^1 dz\int_0^{1-z}dy\left(-\left[\ln\left(1+\frac{\Lambda^2}{\Delta_0}\right)+\frac{\Delta_0(4\Lambda^2+3\Delta_0)}{2(\Lambda^2+\Delta_0)^2}-\frac{3}{2}\right]\right. \\
&\left.-((1-4z+z^2)m^2)\frac{\Lambda^4}{\Delta_0(\Lambda^2+\Delta_0)^2}\right) \\
=&\frac{e^2}{8\pi^2}\int_0^1 dz(1-z)\left(-\left[\ln\left(1+\frac{\Lambda^2}{\Delta_0}\right)+\frac{\Delta_0(4\Lambda^2+3\Delta_0)}{2(\Lambda^2+\Delta_0)^2}-\frac{3}{2}\right]\right. \\
&\left.-((1-4z+z^2)m^2)\frac{\Lambda^4}{\Delta_0(\Lambda^2+\Delta_0)^2}\right) \\
\xrightarrow{\Gamma\to\infty}&\frac{e^2}{8\pi^2}\int_0^1 dz(1-z)\left(-\ln\left(1+\frac{\Lambda^2}{\Delta_0}\right)+\frac{3}{2}-\frac{(1-4z+z^2)m^2}{\Delta_0}\right)
\end{aligned} \tag{33}$$

其中 $\Delta_0=(1-z)^2m^2+z\mu^2=\Delta_m$ 。与公式（24）对比发现，$\delta_1\neq\delta_2$，即 $Z_1\neq Z_2$。这是因为对光子的动量截断会破坏局域规范对称性。

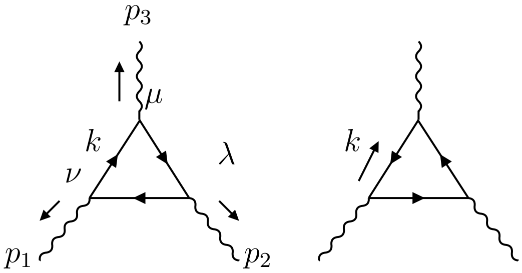
习题 3

问题（a）

在维数正规化下计算 QED 单圈图水平的三光子关联函数 $\langle\Omega\,|A_\mu(x)A_\nu(y)A_\rho(z)|\,\Omega\rangle$ 。一般性的，证明任意奇数个光子的关联函数为零。

解：

该关联函数对应费曼图，



即

$$
\begin{aligned}
\mathrm{i}\Gamma^{(3)} =& (-\mathrm{i}e)^n \int \frac{\mathrm{d}^d k}{(2\pi)^d}(-1)\left\{\mathrm{tr}\left[\gamma^\mu \frac{\mathrm{i}}{\not k - m}\gamma^\nu \frac{\mathrm{i}}{\not k + \not p_1 - m}\gamma^\lambda \frac{\mathrm{i}}{\not k + \not p_1 + \not p_2 - m}\right]\right. \\
&\left. + \,\mathrm{tr}\left[\gamma^\mu \frac{\mathrm{i}}{-(\not k + \not p_1 + \not p_2) - m}\gamma^\lambda \frac{\mathrm{i}}{-(\not k + \not p_1) - m}\gamma^\nu \frac{\mathrm{i}}{-(\not k) - m}\right]\right\} \\
=& (-\mathrm{i}e)^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d}(-1)\left\{\mathrm{tr}\left[\gamma^\mu \frac{\mathrm{i}}{\not k - m}\gamma^\nu \frac{\mathrm{i}}{\not k + \not p_1 - m}\gamma^\lambda \frac{\mathrm{i}}{\not k + \not p_1 + \not p_2 - m}\right]\right. \\
&\left. + \,\mathrm{tr}\left[\gamma^\mu \frac{\mathrm{i}}{\not k + \not p_1 + \not p_2 - m}\gamma^\lambda \frac{\mathrm{i}}{\not k + \not p_1 - m}\gamma^\nu \frac{\mathrm{i}}{\not k - m}\right]^T\right\} \\
=& (-\mathrm{i}e)^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d}(-1)\left\{\mathrm{tr}\left[\gamma^\mu \frac{\mathrm{i}}{\not k - m}\gamma^\nu \frac{\mathrm{i}}{\not k + \not p_1 - m}\gamma^\lambda \frac{\mathrm{i}}{\not k + \not p_1 + \not p_2 - m}\right]\right. \\
&- \,\mathrm{tr}\left[\gamma^0\gamma^2 \frac{\mathrm{i}}{\not k - m}\gamma^0\gamma^2\gamma^0\gamma^2\gamma^\nu\gamma^0\gamma^2\gamma^0\gamma^2 \frac{\mathrm{i}}{\not k + \not p_1 - m}\gamma^0\gamma^2\gamma^0\gamma^2\gamma^\lambda\gamma^0\gamma^2\gamma^0\gamma^2\right. \\
&\left.\left. \cdot\frac{\mathrm{i}}{\not k + \not p_1 + \not p_2 - m}\gamma^0\gamma^2\gamma^0\gamma^2\gamma^\mu\gamma^0\gamma^2\right]\right\} \\
=& (-\mathrm{i}e)^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d}(-1)\left\{\mathrm{tr}\left[\gamma^\mu \frac{\mathrm{i}}{\not k - m}\gamma^\nu \frac{\mathrm{i}}{\not k + \not p_1 - m}\gamma^\lambda \frac{\mathrm{i}}{\not k + \not p_1 + \not p_2 - m}\right]\right. \\
&\left. - \,\mathrm{tr}\left[\gamma^\mu \frac{\mathrm{i}}{\not k - m}\gamma^\nu \frac{\mathrm{i}}{\not k + \not p_1 - m}\gamma^\lambda \frac{\mathrm{i}}{\not k + \not p_1 + \not p_2 - m}\right]\right\} \\
=& 0
\end{aligned}
\tag{34}
$$

类似地，对于任意奇数个光子的关联函数也会在单圈水平两两相消，

$$
\begin{aligned}
\mathrm{i}\Gamma^{(n)} =& (-\mathrm{i}e)^3 \int \sum_{\{\mu\}} \frac{\mathrm{d}^d k}{(2\pi)^d}(-1)\left\{\mathrm{tr}\left[\frac{\mathrm{i}}{\not k - m}\gamma^{\mu_i} \frac{\mathrm{i}}{\not k + \not p_i - m}\gamma^{\mu_j} \frac{\mathrm{i}}{\not k + \not p_i + \not p_j - m}\cdots\gamma^{\mu_k}\right]\right. \\
&\left. + \,\mathrm{tr}\left[\gamma^{\mu_k}\cdots \frac{\mathrm{i}}{-(\not k + \not p_i + \not p_j) - m}\gamma^{\mu_j} \frac{\mathrm{i}}{-(\not k + \not p_i) - m}\gamma^{\mu_i} \frac{\mathrm{i}}{-(\not k) - m}\right]\right\} \\
=& 0
\end{aligned}
\tag{35}
$$

其中 $\sum_{\{\mu\}}$ 为对所有有序对 $\{(\mu_i,\mu_j,\cdots,\mu_k)|(\mu_i,\mu_j,\cdots,\mu_k)\simeq(\mu_j,\cdots,\mu_k,\mu_i)\}$ 除去轮转的等价类求和。

问题（b）

在维数正规化下计算 $\gamma(p_1)\gamma(p_2)\to\gamma(p_3)\gamma(p_4)$ 的散射振幅。仅需判断其是否有限，不需计算出有限项的具体表达式。

解：

$$
\begin{aligned}
\mathrm{i}\Gamma^{(4)} \xrightarrow{\text{发散部分}} (-\mathrm{i}e)^4 \int \frac{\mathrm{d}^d k}{(2\pi)^d}(-1)\frac{1}{(k^2)^4}(&\mathrm{tr}[\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\rho \not{k}\gamma^\sigma \not{k}] + \mathrm{tr}[\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\sigma \not{k}\gamma^\rho \not{k}] \\
&+ \mathrm{tr}[\gamma^\mu \not{k}\gamma^\rho \not{k}\gamma^\nu \not{k}\gamma^\sigma \not{k}] + \mathrm{tr}[\gamma^\sigma \not{k}\gamma^\rho \not{k}\gamma^\nu \not{k}\gamma^\mu \not{k}] + \mathrm{tr}[\gamma^\rho \not{k}\gamma^\sigma \not{k}\gamma^\nu \not{k}\gamma^\mu \not{k}] + \mathrm{tr}[\gamma^\sigma \not{k}\gamma^\nu \not{k}\gamma^\rho \not{k}\gamma^\mu \not{k}])
\end{aligned}
\tag{36}
$$

其中

$$
\begin{aligned}
\mathrm{tr}[\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\rho \not{k}\gamma^\sigma \not{k}] =& 8dk^\mu k^\nu k^\rho k^\sigma - 2dk^2\left(k^\mu k^\nu g^{\rho\sigma} + k^\rho k^\sigma g^{\mu\nu} + k^\mu k^\sigma g^{\nu\rho} + k^\nu k^\rho g^{\mu\sigma}\right) \\
&+ d\left(k^2\right)^2\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)
\end{aligned}
\tag{37}
$$

在圈积分中，$k^\mu k^\nu \to k^2 g^{\mu\nu}/d,\ k^\mu k^\nu k^\rho k^\sigma \to k^4(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\rho\nu})/(d(d+2))$。于是有

$$
\begin{aligned}
\mathrm{tr}\left[\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\rho \not{k}\gamma^\sigma \not{k}\right] &\Rightarrow \left(\frac{8}{d+2}+d-4\right)\left(k^2\right)^2\left(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\rho}\right) + \left(\frac{8}{d+2}-d\right)\left(k^2\right)^2 g^{\mu\rho}g^{\nu\sigma} \\
&\overset{d\to 4}{\Rightarrow} \frac{4}{3}\left(k^2\right)^2\left(g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)
\end{aligned}
\tag{38}
$$

其中 $d\to 4$ 是安全的（发散部分是对数发散）, 其他几项同理，最后可见所有发散部分和为 0.

习题 4

考虑一个赝标量的 Yuakwa 理论：

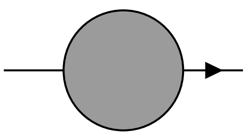
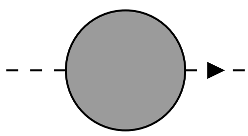
$$
\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 + \bar{\psi}\left(i\gamma^\mu\partial^\mu - M_0\right)\psi_0 - ig_0\bar{\psi}_0\gamma^5\psi_0\phi_0
\tag{39}
$$

问题（a）

对其做微扰重整化，给出重整化后的费曼规则，并在在壳重整化方案下计算所有的抵消项（仅需给出抵消项中的发散部分即可）。

解：

易知，

 $= \frac{iZ_\psi}{\not{p}-M} + \cdots$  $= \frac{iZ_\phi}{q^2-m^2} + \cdots$

于是，可以通过 $\phi_0 \to Z_\phi^{1/2}\phi,\psi_0 \to Z_\psi^{1/2}\psi$ 将 Z_ϕ 和 Z_ψ 吸收入 $\mathcal{L}$，得到

$$\mathcal{L} = \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 + Z_\psi\bar{\psi}\left(i\gamma^\mu\partial^\mu - M_0\right)\psi - ig_0Z_\psi Z_\phi^{1/2}\bar{\psi}\gamma^5\psi\phi \tag{40}$$

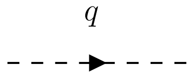
引入物理耦合常数如 $g_0Z_\psi Z_\phi^{1/2} = gZ_g$，和物理质量 m,M, 并将拉格朗日量拆分为两部分则可以得到

$$\begin{aligned}\mathcal{L} =&\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + Z_\psi\bar{\psi}\left(i\gamma^\mu\partial^\mu - M\right)\psi - ig\bar{\psi}\gamma^5\psi\phi \\ &+ \frac{1}{2}\delta_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta_m\phi^2 + \bar{\psi}\left(i\delta_\psi\gamma^\mu\partial^\mu - \delta_M\right)\psi - ig\delta_g\bar{\psi}\gamma^5\psi\phi\end{aligned} \tag{41}$$

其中，

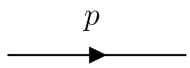
$$\begin{gathered}\delta_\phi = Z_\phi - 1, \quad \delta_\psi = Z_\psi - 1, \\ \delta_m = Z_\phi m_0^2 - m^2, \quad \delta_M = Z_\phi M_0 - M, \\ \delta_g = Z_g - 1 = g_0/gZ_\psi Z_\phi^{1/2} - 1.\end{gathered} \tag{42}$$

那么费曼规则易得，

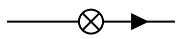


$= \frac{i}{q^2-m^2+i\epsilon}$

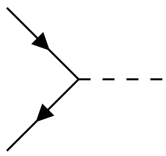
$= i\left(q^2\delta_\phi - \delta_m\right)$



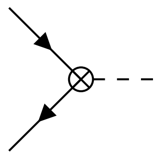
$= \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$



$= i\left(\not{p}\delta_\psi - \delta_M\right)$

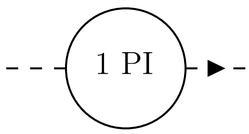


$= -ig\gamma^5$

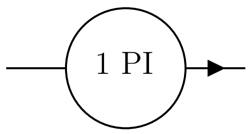


$= -ig\gamma^5\delta_g$

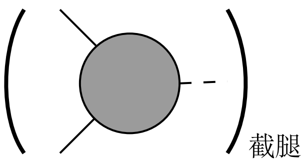
若采取记号，



$= -i\ M^2\left(q^2\right)$



$= -i\ \Sigma\left(\not{p}\right)$



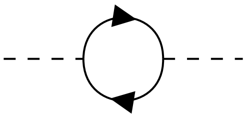
$= -i\ g\Gamma^5(p,p')$

则在壳重整化条件可以写为，

$$\begin{gathered}M^2(p^2)|_{p^2=m^2} = 0, \quad \frac{d}{dp^2}M^2(p^2)|_{p^2=m^2} = 0 \\ \Sigma(\not{p} = M) = 0, \quad \frac{d}{d\not{p}}\Sigma(\not{p})|_{\not{p}=M} = 0 \\ -ig\Gamma^5(p'-p) = -ig\gamma^5\end{gathered} \tag{43}$$

下面计算抵消项，其中 M^2 单圈贡献为

其中第一部分，



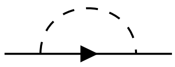
$$
\begin{aligned}
&-(-\mathrm{i}g)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\mathrm{tr}\left[\frac{\mathrm{i}}{\not k-M}\gamma^5\frac{\mathrm{i}}{(\not k-\not p)-M}\gamma^5\right]\\
&=-(-\mathrm{i}g)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{tr}\left[(\not k+M)\gamma^5((\not k-\not p)+M)\gamma^5\right]}{(k^2-M^2)((k-p)^2-M^2)}\\
&=-(-\mathrm{i}g)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{d(k\cdot p-k^2+M^2)}{(k^2-M^2)((k-p)^2-M^2)}\\
&\overset{UV}{\sim}-(-\mathrm{i}g)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{d(k\cdot p-k^2+M^2)}{(k^2-M^2)^2}\left(1+\frac{2k\cdot p}{k^2-M^2}\right)\\
&\overset{UV}{\sim}-(-\mathrm{i}g)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}d\left(-\frac{1}{(k^2-M^2)}+\frac{2(k\cdot p)^2}{(k^2-M^2)^3}\right)\\
&=-(-\mathrm{i}g)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}d\left(-\frac{1}{(k^2-M^2)}+\frac{2p^2}{d}\frac{1}{(k^2-M^2)^2}\right)\\
&\sim\frac{4\mathrm{i}g^2\,(p^2-2M^2)}{(4\pi)^2}\frac{1}{\epsilon}
\end{aligned}
\tag{44}
$$

所以，

$$
\delta_m\sim\frac{-8g^2M^2}{(4\pi)^2}\frac{1}{\epsilon},\quad \delta_\phi=\frac{-4g^2}{(4\pi)^2}\frac{1}{\epsilon}
\tag{45}
$$

对于 Σ 而言有，

其中，

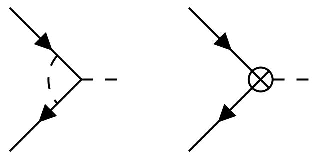


$$
\begin{aligned}
&=g^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\gamma^5\frac{\mathrm{i}}{\not k-M}\gamma^5\frac{\mathrm{i}}{(k-p)^2-m^2}\\
&=g^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-\gamma^5(\not k+M)\gamma^5}{(k^2-M^2)((k-p)^2-m^2)}\\
&=g^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{(\not k-M)}{(k^2-M^2)((k-p)^2-m^2)}\\
&\overset{UV}{\sim}g^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{(\not k-M)}{(k^2-M^2)^2}\left(1+\frac{2p\cdot k}{k^2-M^2}\right)\\
&\overset{UV}{\sim}g^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\left(\frac{-M}{(k^2-M^2)^2}+\frac{2}{d}\frac{\not p}{(k^2-M^2)^2}\right)\\
&\sim\frac{\mathrm{i}g^2(\not p-2M)}{(4\pi)^2}\frac{1}{\epsilon},
\end{aligned}
\tag{46}
$$

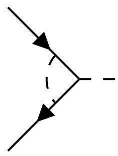
所以，

$$\delta_M \sim \frac{-2g^2M}{(4\pi)^2}\frac{1}{\epsilon}, \quad \delta_\psi \sim \frac{-g^2}{(4\pi)^2}\frac{1}{\epsilon} \tag{47}$$

最后，对于 Γ^5 而言有



其中，



$$\begin{aligned}
&= g^3\int\frac{\mathrm{d}^dk}{(2\pi)^d}\gamma^5\frac{\mathrm{i}}{\not{k}-M}\gamma^5\frac{\mathrm{i}}{\not{k}-M}\gamma^5\frac{\mathrm{i}}{k^2-m^2}\\
&= -ig^3\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\gamma^5(\not{k}+M)\gamma^5(\not{k}+M)\gamma^5}{(k^2-M^2)^2(k^2-m^2)}\\
&= ig^3\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\gamma^5}{(k^2-M^2)(k^2-m^2)}\\
&\overset{UV}{\sim} ig^3\gamma^5\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k^2-M^2)^2}\\
&\sim -\frac{g^3\gamma^5}{(4\pi)^2}\frac{2}{\epsilon}
\end{aligned} \tag{48}$$

所以，

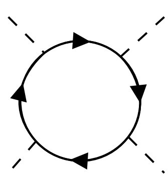
$$\delta_g \sim \frac{2g^3}{(4\pi)^2}\frac{1}{\epsilon} \tag{49}$$

问题（b）

在这个理论中计算 $\phi\phi \to \phi\phi$ 的单圈散射振幅。证明这个振幅是紫外发散的，且发散行为无法被已有的抵消项抵消。为了使这个理论有意义，需要在拉氏量中引入新的一项 $\Delta\mathcal{L} = -\frac{\lambda_0}{4!}\phi_0^4$ 并对其作重整化。这个计算说明了一个一般事实：四维场论中量纲小于等于四的算符，如没有对称性保护，终究会在量子修正下出现，也就是在量子场论中，If something CAN happen, it will happen.

解：

散射振幅为，

$i\mathcal{M} =$ 

$$= (-1)g^4\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\mathrm{tr}\left[\left(\gamma^5\frac{\mathrm{i}}{\not{k}-M}\right)^4\right] = (-1)g^4\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{tr}[\mathbb{I}]}{(\not{k}-M)^2} \sim -\frac{8\mathrm{i}g^4}{(4\pi)^2}\frac{1}{\epsilon} \tag{50}$$

即该振幅是紫外发散的，需要引入形如 $\delta_\lambda\phi^4$ 的抵消项，即在拉氏量中引入 $\Delta\mathcal{L} = -\frac{\lambda_0}{4!}\phi_0^4$。