第三、四周作业参考答案

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习题 1

证明时间演化算符

$$U(t,t_0)=\exp\left[-i\int_{t_0}^{t}dt'H_I(t)\right] \tag{1}$$

满足如下关系：

$$U(t,t_1)U(t_1,t_2)=U(t,t_2),\quad U(t,t_1)U^{\dagger}(t_2,t_1)=U(t,t_2) \tag{2}$$

解：

时序算符

$$U(t,t_1)=T\left\{\exp\left(-i\int_{t_1}^{t}dt'H_I(t')\right)\right\} \tag{3}$$

是下列一阶方程的解

$$i\frac{\partial}{\partial t}U(t,t_1)=H_I(t)U(t,t_1) \tag{4}$$

因为

$$\begin{aligned}
i\frac{\partial}{\partial t}U(t,t_1)&=i\frac{\partial}{\partial t}\sum_{n=0}^{\infty}(-i)^n\int_{t_1}^{t}dt_1'\int_{t_1}^{t_1'}dt_2'\cdots\int_{t_1}^{t_{n-1}'}dt_n'H_I(t_1')H_I(t_2')\cdots H_I(t_n')\\
&=H_I(t)\sum_{n=1}^{\infty}(-i)^{n-1}\int_{t_1}^{t}dt_2'\cdots\int_{t_1}^{t_{n-1}'}dt_n'H_I(t_2')\cdots H_I(t_n')\\
&=H_I(t)\sum_{n=1}^{\infty}(-i)^{n-1}\int_{t_1}^{t}dt_1'\cdots\int_{t_1}^{t_{n-2}'}dt_{n-1}'H_I(t_1')\cdots H_I\left(t_{n-1}'\right)\\
&=H_I(t)U(t,t_1).
\end{aligned} \tag{5}$$

记 $U_S(t,t_1)\equiv e^{-iH_0(t-t_0)}U(t,t_1)e^{iH_0(t_1-t_0)}$ 且由定义 $H_I(t)=e^{iH_0(t-t_0)}(H-H_0)e^{-iH_0(t-t_0)}$，代入（4）得

$$\begin{aligned}
i\frac{\partial}{\partial t}\left(e^{iH_0(t-t_0)}U_S(t,t_1)e^{-iH_0(t_1-t_0)}\right)&=e^{iH_0(t-t_0)}(H-H_0)e^{iH_0(t-t_0)}e^{iH_0(t-t_0)}U_S(t,t_1)e^{-iH_0(t_1-t_0)}\\
\Rightarrow -H_0U_S(t,t_1)+i\frac{\partial}{\partial t}U_S(t,t_1)&=(H-H_0)U_S(t,t_1)\\
\Rightarrow i\frac{\partial}{\partial t}U_S(t,t_1)&=HU_S(t,t_1)
\end{aligned} \tag{6}$$

因此

$$U_S(t,t_1)=e^{-iH(t-t_0)} \tag{7}$$

则

$$U(t,t_1)=e^{iH_0(t-t_0)}e^{-iH(t-t_0)}e^{-iH_0(t_1-t_0)} \tag{8}$$

其也满足边界条件 $U(t_1,t_1)=1$, 易由此边界条件和一阶线性微分方程（4）解唯一性确认（8）与（3）等价。那么，

$$\begin{aligned}U(t,t_1)U(t_1,t_2)=&e^{iH_0(t-t_0)}e^{-iH(t-t_1)}e^{-iH_0(t_1-t_0)}e^{iH_0(t_1-t_0)}e^{-iH(t_1-t_2)}e^{-iH_0(t_2-t_0)}\\=&e^{iH_0(t-t_0)}e^{-iH(t-t_2)}e^{-iH_0(t_2-t_0)}\\=&U(t,t_2),\end{aligned} \tag{9}$$

$$\begin{aligned}U(t,t_1)U^\dagger(t_2,t_1)=&e^{iH_0(t-t_0)}e^{-iH(t-t_1)}e^{-iH_0(t_1-t_0)}\left[e^{iH_0(t_2-t_0)}e^{-iH(t_2-t_1)}e^{-iH_0(t_1-t_0)}\right]^\dagger\\=&e^{iH_0(t-t_0)}e^{-iH(t-t_2)}e^{-iH_0(t_2-t_0)}\\=&U(t,t_2).\end{aligned} \tag{10}$$

习题 2

证明编时乘积 $T\{\phi(x_1)\phi(x_2)\}$ 和正规乘积 $\phi(x_1)\phi(x_2)$ 关于 x_1 和 x_2 的交换对称，从而证明费曼传播子 $\Delta_F(x_1-x_2)$ 也是关于 $x_1\leftrightarrow x_2$ 对称。

解:

由

$$\begin{aligned}T\{\phi(x_1)\phi(x_2)\}=\theta\left(x_1^0-x_2^0\right)\phi(x_1)\phi(x_2)+\theta\left(x_2^0-x_1^0\right)\phi(x_2)\phi(x_1),\\T\{\phi(x_2)\phi(x_1)\}=\theta\left(x_2^0-x_1^0\right)\phi(x_2)\phi(x_1)+\theta\left(x_1^0-x_2^0\right)\phi(x_1)\phi(x_2).\end{aligned} \tag{11}$$

得

$$T\{\phi(x_1)\phi(x_2)\}=T\{\phi(x_2)\phi(x_1)\} \tag{12}$$

记

$$\phi^+(x)=\int\frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{p}}}a_\mathbf{p}e^{-ipx},\quad \phi^-(x)=\int\frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2\omega_\mathbf{p}}}a_\mathbf{p}^\dagger e^{ipx}, \tag{13}$$

则 $\phi(x)=\phi^+(x)+\phi^-(x)$。那么

$$\begin{aligned}:\phi(x_1)\phi(x_2):=&:\left(\phi^+(x_1)+\phi^-(x_1)\right)\left(\phi^+(x_2)+\phi^-(x_2)\right):\\=&\phi^+(x_1)\phi^+(x_2)+\phi^-(x_1)\phi^+(x_2)+\phi^-(x_2)\phi^+(x_1)\\&+\phi^-(x_1)\phi^-(x_2)\\=&:\phi(x_2)\phi(x_1):\end{aligned} \tag{14}$$

而费曼传播子

$$\Delta_F(x_1-x_2)=\overbrace{\phi(x_1)\phi}(x_2)=T\{\phi(x_1)\phi(x_2)\}-:\phi(x_1)\phi(x_2): \tag{15}$$

所以

$$\Delta_F(x_1-x_2)=\Delta_F(x_2-x_1). \tag{16}$$

习题 3

通过具体计算验证如下 Wick 公式：

$$
\begin{aligned}
T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\right\} = &:\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right): + \phi\left(x_1\right)\Delta_F\left(x_2-x_3\right) \\
&+\phi\left(x_2\right)\Delta_F\left(x_3-x_1\right)+\phi\left(x_3\right)\Delta_F\left(x_1-x_2\right)
\end{aligned}
\tag{17}
$$

其中 $\phi(x)$ 是实标量场，$\Delta_F(x)$ 是费曼传播子。

解：

假设 $x_1^0 > x_2^0 > x_3^0$，

$$
\begin{aligned}
&T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\right\} \\
=&\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right) \\
=&\left(\phi^+\left(x_1\right)+\phi^-\left(x_1\right)\right)\left(\phi^+\left(x_2\right)+\phi^-\left(x_2\right)\right)\left(\phi^+\left(x_3\right)+\phi^-\left(x_3\right)\right) \\
=&\phi^+\left(x_1\right)\phi^+\left(x_2\right)\phi^+\left(x_3\right)+\phi^+\left(x_1\right)\phi^+\left(x_2\right)\phi^-\left(x_3\right)+\phi^+\left(x_1\right)\phi^-\left(x_2\right)\phi^+\left(x_3\right)+\phi^+\left(x_1\right)\phi^-\left(x_2\right)\phi^-\left(x_3\right) \\
&+\phi^-\left(x_1\right)\phi^+\left(x_2\right)\phi^+\left(x_3\right)+\phi^-\left(x_1\right)\phi^+\left(x_2\right)\phi^-\left(x_3\right)+\phi^-\left(x_1\right)\phi^-\left(x_2\right)\phi^+\left(x_3\right)+\phi^-\left(x_1\right)\phi^-\left(x_2\right)\phi^-\left(x_3\right) \\
=&:\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right):+\phi^-\left(x_1\right)\left[\phi^+\left(x_2\right),\phi^-\left(x_3\right)\right]+\left[\phi^+\left(x_1\right),\phi^-\left(x_2\right)\right]\phi^+\left(x_3\right) \\
&+\left[\phi^+\left(x_1\right),\phi^-\left(x_2\right)\phi^-\left(x_3\right)\right]+\left[\phi^+\left(x_1\right)\phi^+\left(x_2\right),\phi^-\left(x_3\right)\right] \\
=&:\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right):+\phi^-\left(x_1\right)\left[\phi^+\left(x_2\right),\phi^-\left(x_3\right)\right]+\left[\phi^+\left(x_1\right),\phi^-\left(x_2\right)\right]\phi^+\left(x_3\right) \\
&+\phi^-\left(x_2\right)\left[\phi^+\left(x_1\right),\phi^-\left(x_3\right)\right]+\left[\phi^+\left(x_1\right),\phi^-\left(x_2\right)\right]\phi^-\left(x_3\right)+\phi^+\left(x_1\right)\left[\phi^+\left(x_2\right),\phi^-\left(x_3\right)\right] \\
&+\left[\phi^+\left(x_1\right),\phi^-\left(x_3\right)\right]\phi^+\left(x_2\right) \\
=&:\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right):+\phi\left(x_1\right)\overbrace{\phi\left(x_2\right)\phi}\left(x_3\right)+\overbrace{\phi\left(x_1\right)\phi\left(x_2\right)\phi}\left(x_3\right)+\overbrace{\phi\left(x_1\right)\phi}\left(x_2\right)\phi\left(x_3\right) \\
=&:\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right):+\phi\left(x_1\right)\Delta_F\left(x_2-x_3\right)+\phi\left(x_2\right)\Delta_F\left(x_3-x_1\right)+\phi\left(x_3\right)\Delta_F\left(x_1-x_2\right)
\end{aligned}
\tag{18}
$$

该性质不依赖于假设 $x_1^0 > x_2^0 > x_3^0$, 其他情况同理可证。

习题 4

考虑如下的标量汤川理论：

$$
\mathcal{L} = \partial_\mu\psi\partial^\mu\psi^* - M^2\psi^*\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - g\psi^*\psi\phi \tag{19}
$$

其中 ψ 是复标量核子场，ϕ 是实标量介子场。画出如下过程的相关费曼图并计算散射振幅：

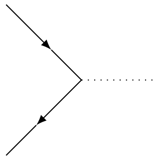
问题 (a)

- 核子-反核子散射：$\psi + \psi^* \to \phi$，保留到 $\mathcal{O}(g)$

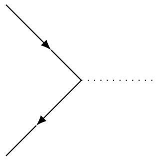
解：

 $= \frac{i}{p^2-M^2+i\epsilon}$

 $= \frac{i}{p^2-m^2+i\epsilon}$

 $= -ig$

核子-反核子一阶散射振幅为

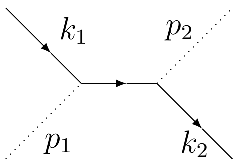
$i\mathcal{M} =$  $= -ig$

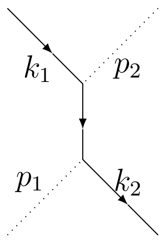
问题（b）

- 核子-介子散射：$\psi\phi \to \psi\phi$，保留到 $\mathcal{O}\left(g^2\right)$

解：

散射振幅为

$i\mathcal{M} =$ 

$+$ 

$$\begin{aligned} &= -ig^2\left[\frac{1}{\left(k_1+p_1\right)^2-M^2+i\epsilon}+\frac{1}{\left(p_1-k_2\right)^2-M^2+i\epsilon}\right] \\ &= -ig^2\left(\frac{1}{s-M^2+i\epsilon}+\frac{1}{t-M^2+i\epsilon}\right) \end{aligned} \tag{20}$$

习题 5

康普顿散射描述了高能光子与初始静止的电子散射的过程：

$$\gamma\left(k_1\right)+e\left(p_1\right) \to \gamma\left(k_2\right)+e\left(p_2\right) \tag{21}$$

该过程的散射振幅如何计算将在以后的课程中介绍，这里我们只需要知道散射振幅的表达式（已对所有末态自旋求和以及初态自旋求平均）：

$$\begin{aligned} |M|^2 =& 32\pi^2\alpha^2\left[\frac{m^4+m^2(3s+u)-su}{\left(m^2-s\right)^2}+\frac{m^4+m^2(3u+s)-su}{\left(m^2-u\right)^2}\right. \\ &\left.+\frac{2m^2\left(s+u+2m^2\right)}{\left(m^2-s\right)\left(m^2-u\right)}\right]+\mathcal{O}\left(\alpha^4\right) \end{aligned} \tag{22}$$

其中 $\alpha = 1/137$ 是精细结构常数，m 是电子质量，$s=\left(k_1+p_1\right)^2, u=\left(k_1-p_2\right)^2$。

问题 (a)

- 将 Mandelstam 变量 s 和 u 用入射和出射光子的能量 ω 和 ω' 表示。

解：

$$s = m^2 + 2m\omega \tag{23}$$
$$u = m^2 - 2m\omega' \tag{24}$$

问题 (b)

- 将出射光子相对入射光子的散射角 θ 用 ω 和 ω' 表示。

解：

$$u = m^2 - 2k_1^\mu p_{2\mu} = m^2 - 2m\omega + 2\omega\omega'(1-\cos\theta) \tag{25}$$
$$\theta = \arccos\left[1 - 2m\left(\frac{1}{\omega'} - \frac{1}{\omega}\right)\right] \tag{26}$$

问题 (c)

- 计算关于出射光子方位角 $d\Omega = \sin\theta d\theta d\phi$ 的微分散射截面，并将其表达为 ω 和 ω' 的函数。你的结果应与 Klein-Nishina 公式一致：

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}\frac{\omega'^2}{\omega^2}\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right) \tag{27}$$

解：

代入上述结果得

$$|\mathcal{M}|^2 = 32\pi^2\alpha^2\left[\frac{4\left[m^2 + m(\omega-\omega') + \omega\omega'\right]}{\omega^2} + \frac{4\left(m^2 + -m\omega' + \omega\omega'\right)}{\omega'^2} - \frac{4m^2 + 2m(\omega-\omega')}{\omega\omega'}\right] \tag{28}$$

而

$$\begin{aligned}\int d\sigma =& \frac{1}{2E_{k_1}2E_{p_1}\left|\mathbf{v}_{k_1} - \mathbf{v}_{p_1}\right|}\int \frac{d^3\mathbf{k}_2}{(2\pi)^3}\frac{1}{2E_{\mathbf{k}_2}}\frac{d^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}_2}}(2\pi)^4\delta^{(4)}\left(p_2 + k_2 - p_1 - k_1\right)|\mathcal{M}|^2 \\ =& \frac{1}{64\pi^2 m\omega}\int d\Omega \frac{\omega'}{|m + \omega(1-\cos\theta)|}|\mathcal{M}|^2\end{aligned} \tag{29}$$

于是

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} =& \frac{\alpha^2}{2m\omega}\frac{\omega'}{|m + \omega(1-\cos\theta)|}\left[\frac{4\left[m^2 + m(\omega-\omega') + \omega\omega'\right]}{\omega^2} + \frac{4\left(m^2 + -m\omega' + \omega\omega'\right)}{\omega'^2}\right. \\ & \left. - \frac{4m^2 + 2m(\omega-\omega')}{\omega\omega'}\right] \\ =& \frac{\alpha^2}{2m^2}\frac{\omega'^2}{\omega^2}\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right)\end{aligned} \tag{30}$$

习题 6

量子非谐振子可以看成 0+1 维（零维空间，一维时间）的量子场论，其拉格朗日量如下：

$$L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2 - \frac{1}{4!}\lambda\phi^4 \tag{31}$$

正则对易关系为：

$$[\phi, \dot{\phi}] = i \tag{32}$$

我们这里感兴趣的是基态能量 E_0，可以写成关于 λ 展开的级数：

$$E_0 = \frac{\omega}{2} + \sum_{n=1}^{\infty} \lambda^n A_n(m) \tag{33}$$

其中第一项是简谐振子的零点能，后面的项对应相互作用对简谐振子的微扰展开。

问题 (a)

- 确定 ϕ, ω, λ 的质量量纲

解：

$$[\phi] = [M]^{-\frac{1}{2}}$$
$$[\omega] = [M]^1$$
$$[\lambda] = [M]^3$$

问题 (b)

- 证明 E_0 可通过计算费曼图得到

$$E_0 = \frac{\omega}{2} + \sum_n \sum_i \frac{F_{n,i}}{\mathrm{sym}_{n,i}} \tag{34}$$

其中 $F_{n,i}$ 是 n 阶图 $\mathcal{O}(\lambda^n)$ 中第 i 个独立的连通真空费曼图的贡献，$\mathrm{sym}_{n,i}$ 是 n 阶图中第 i 个连通真空图形的对称因子。

解：

记 $\lambda = 0$ 时，真空为 $|0\rangle$, 记 $H_i = \frac{1}{4!}\lambda\phi^4$, $H_0 = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\omega^2\phi^2$ 则

$$e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle \tag{35}$$

当 $T \to \infty(1-i\epsilon)$, 高激发态指数压低得

$$\begin{aligned}
\lim_{T\to\infty(1-i\epsilon)} e^{-iHT}|0\rangle &= \lim_{T\to\infty(1-i\epsilon)} e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle \\
\Rightarrow \lim_{T\to\infty(1-i\epsilon)} \langle 0|e^{-iH2T}|0\rangle &= \lim_{T\to\infty(1-i\epsilon)} e^{-iE_02T}\langle 0|\Omega\rangle\langle\Omega|0\rangle \\
\Rightarrow \lim_{T\to\infty(1-i\epsilon)} \langle 0|e^{-iH_0(T-t_0)}U(T,-T)e^{+iH_0(-T-t_0)}|0\rangle &= \lim_{T\to\infty(1-i\epsilon)} e^{-iE_02T}|\langle 0|\Omega\rangle|^2 \\
\Rightarrow \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,-T)|0\rangle &= \lim_{T\to\infty(1-i\epsilon)} e^{-i(E_0-\frac{\omega}{2})2T}|\langle 0|\Omega\rangle|^2
\end{aligned} \tag{36}$$

其中 $U(T,-T)=T\left\{\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]\right\}$ 其中 $H_I(t)=e^{iH_0(t-t_0)}H_ie^{-iH_0(t-t_0)}$。而

$$\langle 0|U(T,-T)|0\rangle=\exp(-i\sum_n\sum_i\frac{F_{n\,i}}{sym_{n,i}}2T) \tag{37}$$

在时间极限和微扰意义下对比 (36)(37) 有时间依赖的相位得到

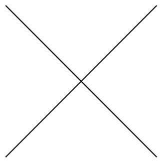
$$E_0=\frac{\omega}{2}+\sum_n\sum_i\frac{F_{n,i}}{\mathrm{sym}_{n,i}} \tag{38}$$

问题 (c)

- 写下上述理论的动量空间费曼规则。

解:

 $=\frac{i}{(p^0)^2-\omega^2}$

 $=-i\lambda$

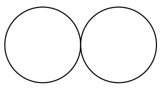
$$D_F(t,t')=\overline{\phi(t)\phi}(t')=\frac{e^{-i\omega|t-t'|}}{2\omega}=\int\frac{dp^0}{2\pi i}\frac{-1}{(p^0)^2-\omega^2}e^{-ip^0(t-t')}$$

问题 (d)

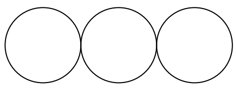
- 画出直到 $n=4$ 的所有连通真空图形，并求出相应对称因子。

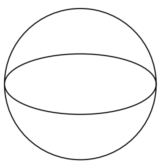
解:

n = 1

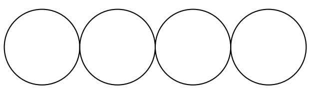
 8

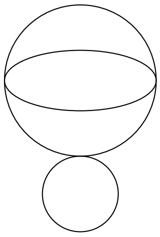
n = 2

 16

 48

n = 3

 32



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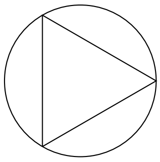

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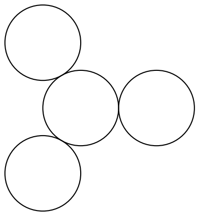
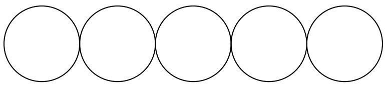


48

n = 4



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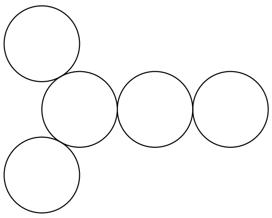

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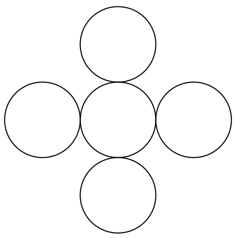

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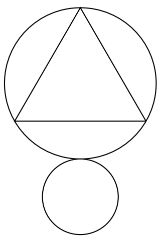

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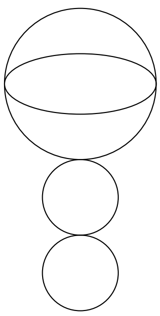

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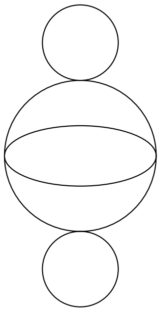

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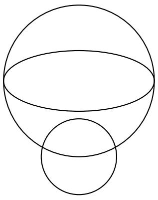

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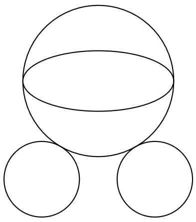

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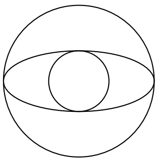

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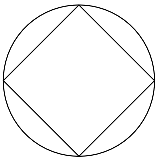

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问题 (e)

- 计算 E_n 的前两项，即 A_1，A_2。

解：

$$F_{1,1} = \lambda \left(\int \frac{dp^0}{2\pi} \frac{-1}{(p^0)^2 - \omega^2 + i\epsilon} \right)^2 = \frac{\lambda}{(2\omega)^2} \tag{39}$$

$$F_{2,1} = -i\lambda^2 \left(\int \frac{dp^0}{2\pi} \frac{-1}{(p^0)^2 - \omega^2 + i\epsilon} \right)^2 \int \frac{dp^0}{2\pi} \left(\frac{-1}{(p^0)^2 - \omega^2 + i\epsilon} \right)^2 = -\frac{\lambda^2}{(2\omega)^2} \frac{1}{4\omega^3} = -\frac{\lambda^2}{16\omega^5} \tag{40}$$

$$
\begin{aligned}
F_{2,2} =& i\lambda^2 \int \frac{dp_1^0}{2\pi} \int \frac{dp_2^0}{2\pi} \int \frac{dp_3^0}{2\pi} \frac{i}{(p_1^0)^2 - \omega^2 + i\epsilon} \frac{i}{(p_2^0)^2 - \omega^2 + i\epsilon} \frac{i}{(p_3^0)^2 - \omega^2 + i\epsilon} \\
&\times \frac{i}{(p_1^0 + p_2^0 + p_3^0)^2 - \omega^2 + i\epsilon} \\
=& \lambda^2 \int \frac{dp_1^0}{2\pi} \int \frac{dp_2^0}{2\pi} \frac{1}{(p_1^0)^2 - \omega^2 + i\epsilon} \frac{1}{(p_2^0)^2 - \omega^2 + i\epsilon} \frac{1}{\omega(p_1 + p_2 - 2\omega + i\epsilon)(p_1 + p_2 + 2\omega - i\epsilon)} \\
=& -i\lambda^2 \int \frac{dp_1^0}{2\pi} \frac{1}{(p_1^0)^2 - \omega^2 + i\epsilon} \frac{3}{4\omega^2((p_1^0)^2 - 9\omega^2 + i\epsilon)} \\
=& \frac{\lambda^2}{32\omega^5}
\end{aligned}
\tag{41}
$$

所以,

$$
A_1 = \frac{1}{32\omega^2} \tag{42}
$$

$$
A_2 = -\frac{1}{16}\frac{1}{16\omega^5} + \frac{1}{48}\frac{1}{32\omega^5} = -\frac{7}{1536\omega^5} \tag{43}
$$