Homework for week 1 and 2

September 16, 2023

1. Show that the 4D volumn element is invariant under Lorentz transformation:
$$d^4x = d^4x' \tag{1}$$

2. Show that under a Lorentz transformation,
$$\frac{d^3k}{2\omega_k} \to \frac{d^3k'}{2\omega'_k} \tag{2}$$
namely, it's also Lorentz invariant.

3. Show that the action for a free Klein-Gordon field is invariant for Lorentz transformation Λ, where $\det(\Lambda) = 1$.

4. Explain why we don't need a linear term of ϕ in the Lagrangian for a free Klein-Gordon field.

5. In d space-time dimension, what is the dimension of the following Lagrangian? What is the dimension of a_n?
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \sum_{n=2}^{\infty} a_n\phi^n \tag{3}$$

6. A space-time translation operator $T(a)$ is defined as $T(a) = \exp(ia^\mu P_\mu)$, where P_μ is the momentum operator. Show that $T(a)$ is unitary. A scalar field transforms under $T(a)$ as $T(a)^\dagger\phi(x)T(a) = \phi(x - a)$.
   - Let a^μ be infinitesimal. Derive an expression for $[P^\mu, \phi(x)]$.
   - Show that the time component of your result is equivalent to the Heisenberg equation of motioni $\dot{\phi} = i[H, \phi]$.
   - For a free field, use the Heisenberg equation to derive the Klein-Gordon equation.
   - Define a spatial momentum operator
   $$\boldsymbol{P} = -\int d^3x\, \pi(x)\boldsymbol{\nabla}\phi(x) \tag{4}$$
   Use the canonical commutation relations to show that $\boldsymbol{P}$ obeys the relation you derived in part (a).

- Express $\boldsymbol{P}$ in terms of $a_{\boldsymbol{k}}$ and $a^\dagger_{\boldsymbol{k}}$.

7. The time-ordered product of two fields, $A(x)$ and $B(y)$, is defined by

$$T[A(x)B(y)] = \theta(x^0 - y^0)A(x)B(y) + \theta(y^0 - x^0)B(y)A(x) \tag{5}$$

Using only the field equation and the equal time commutation relations, show that, for a free scalar field of mass m,

$$(\partial_x^2 + m^2)\langle 0|T[\phi(x)\phi(y)]|0\rangle = c\delta^{(4)}(x-y) \tag{6}$$

Find the constant c.

8. Considering the following Lagrangian:

$$\mathcal{L} = i\psi^*\partial_0\psi + b\nabla\psi^*\cdot\nabla\psi$$

where ψ is a complex scalar field and b is a real constant.

- Find the Euler–Lagrange equations.
- Find the plane-wave solutions, those for which ψ is of the form $\psi = e^{-i\omega t + i\boldsymbol{p}\cdot\boldsymbol{x}}$, and find ω as a function of $\boldsymbol{p}$.
- Although this theory is not Lorentz-invariant, it is invariant under spacetime translations and the internal symmetry transformation

$$\psi \to e^{-i\alpha}\psi\,, \quad \psi^* \to e^{i\alpha}\psi^*$$

Thus it possesses a conserved energy, a conserved linear momentum, and a conserved charge associated with the internal symmetry. Find these quantities as integrals of the fields and their derivatives. Fix the sign of b by demanding the energy be bounded below.

- Canonically quantize the theory. Identify appropriately normalized coefficients in the expansion of the fields in terms of plane wave solutions with annihilation and/or creation operators. Write the energy, linear momentum and internal-symmetry charge in terms of these operators.
- Find the equation of motion for the single particle state $|\boldsymbol{k}\rangle$ and the two particle state $|\boldsymbol{k}_1\boldsymbol{k}_2\rangle$ in the Schordinger Picture. What physical quantities do b and the internal symmetry charge correspond to?